Freshman's dream which is given to the incorrect identity $(x+y)^n = x^n + y^n$. The *sophomore's dream* has a similar too-good-to-be-true feel, but is true. Consider a indefinite integral

$$\int x^x \, dx,$$

which does not have an elementary primitive function.

However, this proposition that definite integral

$$\int_0^1 x^x \, dx = -\sum_{n=1}^{\infty} (-n)^{-n} \quad ,$$

is easy to prove, using an approach as *freshman' dream.* The original proof was given by Bernoulli(1697).

1 Sophomore's Dream

First, use Taylor series to enpand x^x as

$$x^x = e^{x \ln x} = \sum_{n=0}^{\infty} \frac{x^n (\ln x)^n}{n!} \, dx$$

Then, perform variable substitution

$$x = e^{-\frac{u}{n+1}}$$

in the integral, obtain immediately:

$$\begin{aligned} \int_0^1 \sum_{n=0}^{\infty} x^n (\ln x)^n \, dx &= \sum_{n=0}^{\infty} \int_0^1 x^n (\ln x)^n \, dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}} \int_0^{\infty} u^n e^{-u} \, du \end{aligned}$$

Notice that, gamma function

$$\Gamma(n+1) = n! = \int_0^{\infty} u^n e^{-u} \, du.$$

In fact, $\Gamma(n+1)$ is the factorial for integer n.

Finally, we obtain

$$\begin{aligned}
\int_0^1 x^x\,dx &= \int_0^1 \sum_{n=0}^{\infty} \frac{x^n(\ln x)^n}{n!}\,dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}n!} \int_0^{\infty} u^n e^{-u}\,du \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^n} \\
&= -\sum_{n=1}^{\infty} (-n)^{-n} \quad \blacksquare
\end{aligned}$$

Remark1: Freshman's dream is incorrect in general, but correct when one is working in a commutative ring of prime characteristic p with n being a power of p. The correct result in a general commutative context is given by the binomial theorem.

Remark2: The first actual attestation of the phrase *freshman's dream* seems to be in Hungerford's graduate algebra textbook (1974).

这个命题的证明应用了类似"一年生之梦"的想法，将x^x泰勒展开之后逐项计算，居然成功地应用gamma函数的积分形式消去了$n!$和积分项. 这个证明就像梦一般美妙，展开后得到了很好的结果（展开这种有着非初等原函数的积分而不考虑收敛性会带来不幸？），"二年生之梦"应该也是因此得名.

PS: Remark2提及的*Hungerford's graduate algebra textbook*正是这一年学习抽代用的GTM73，真是太巧了$\omega\omega\omega\omega\omega$.